

# Exact Solutions for Boson-Fermion Stars in (2+1) dimensions

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## Abstract

We solve Einstein equations coupled to a complex scalar field with infinitely large self-interaction, degenerate fermions, and a negative cosmological constant in  $(2 + 1)$  dimensions. Exact solutions for static boson-fermion stars are found when circular symmetry is assumed. We find that the minimum binding energy of boson-fermion star takes a negative value if the value of the cosmological constant is sufficiently small.

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## I. INTRODUCTION

Lower-dimensional gravity provides us with a useful arena where we can often construct exactly solvable models and can study the model analytically. The solutions for relativistic boson stars ([1] [2] for reviews) are only numerically obtained in four dimensions. In the previous paper [3], we obtained an exact solution for a nonrotating boson star in  $(2 + 1)$  dimensional gravity with a negative cosmological constant. We assumed that the scalar field has a strong self-interaction. An infinitely large self-interaction term in the model leads to much simplification as in the  $(3 + 1)$  dimensional case [4].

In the present paper, we derive an exact solution for a nonrotating boson-fermion star in  $(2 + 1)$  dimensional gravity with a negative cosmological constant. We assume that the scalar field has a strong self-interaction as in the previous paper.

Four-dimensional boson-fermion stars have been studied by many authors [5]. Our three-dimensional model includes an additional constant, the cosmological constant. We will see that the value of the cosmological constant is restricted by the stability condition of the boson-fermion star. We will also find, nevertheless, that some physical quantities of the boson-fermion star behave as in the four-dimensional case.

In Section II, we show the field equations which we solve. In Section III, we obtain exact solutions for the field equations shown in Section II. We pay our attention on the expression of mass of the boson-fermion star. The particle numbers of bosons and fermions for the boson-fermion star described by the exact solutions is given in Section IV. In Section V, we restrict our attention to the fermion star, i.e., the case that the particle number of bosons vanishes. We discuss stability of the boson-fermion star in Section VI. Section VII is devoted to conclusion.

## II. FIELD EQUATIONS

We consider a complex scalar field  $\varphi$  with mass  $m_B$  and a quartic self-coupling constant  $\lambda$ . We assume that the coupling constant takes an infinitely large value. The equation of motion for the scalar field is

$$\nabla^2 \varphi - m_B^2 \varphi - \lambda |\varphi|^2 \varphi = 0. \quad (1)$$

In addition, we assume that a degenerate fermion field with mass  $m_F$  is coupled to gravity. In this paper, we assume no direct coupling between the boson and fermion fields.

The Einstein equation is written as

$$R_\nu^\mu - \frac{1}{2} \delta_\nu^\mu R = 8\pi G (T_\nu^{B\mu} + T_\nu^{F\mu}) + C \delta_\nu^\mu, \quad (2)$$

where the positive constant  $C$  stands for the (negative) cosmological constant.  $G$  is the Newton constant.

The energy-momentum tensor of the boson field is

$$T_\nu^{B\mu} = 2\text{Re} \left( \nabla^\mu \varphi^* \nabla_\nu \varphi - \frac{1}{2} |\nabla \varphi|^2 \delta_\nu^\mu \right) - m_B^2 |\varphi|^2 \delta_\nu^\mu - \frac{\lambda}{2} |\varphi|^4 \delta_\nu^\mu. \quad (3)$$

The energy-momentum tensor of the fermion field is written in the perfect fluid form as

$$T_\nu^{F\mu} = \text{diag.} (-\rho, P, P), \quad (4)$$

where the hydrostatic equilibrium is assumed.

In three dimensions, the energy density and pressure are given by [6]

$$\rho = \frac{m_F^3}{6\pi} \left( \frac{\mu^3}{m_F^3} - 1 \right), \quad (5)$$

and

$$P = \frac{m_F^3}{12\pi} \left( \frac{\mu^3}{m_F^3} - 3 \frac{\mu}{m_F} + 2 \right), \quad (6)$$

respectively. Here  $\mu$  stands for a chemical potential for the fermion field.

The number density of fermions is given by

$$n_F = \frac{m_F^2}{4\pi} \left( \frac{\mu^2}{m_F^2} - 1 \right). \quad (7)$$

We assume the three-dimensional metric for a static circularly symmetric spacetime as

$$ds^2 = -\alpha^2(r)dt^2 + \beta^2(r)dr^2 + \frac{\gamma^2(r)}{\gamma_0^2}d\theta^2, \quad (8)$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are functions of the radial coordinate  $r$  only. The constant  $\gamma_0$  will be determined later (see Eqs. (35-36) and below.). Because of the coordinate invariance with respect to  $r$ , we have a freedom in choosing the function  $\beta(r)$ . We will use this residual “gauge choice” later.

We also assume that the complex scalar has a phase which is linear in the temporal coordinate:

$$\varphi = e^{-i\omega t}\varphi(r), \quad (9)$$

where  $\varphi(r)$  is a function of the radial coordinate  $r$  only and  $\omega$  is a constant.

For a static circularly symmetric spacetime,  $\rho$  and  $P$  have only dependence on  $r$ . Thus  $\mu$  is also assumed to be a function of the radial coordinate  $r$  only.

Then the Einstein equation (2) implies the following equations:

$$\frac{1}{\beta\gamma} \frac{d}{dr} \left( \frac{1}{\beta} \frac{d\gamma}{dr} \right) = 8\pi G \left[ -\frac{1}{\beta^2} \left( \frac{d\varphi}{dr} \right)^2 - \frac{\omega^2}{\alpha^2} \varphi^2 - m_B^2 \varphi^2 - \frac{\lambda}{2} \varphi^4 - \rho \right] + C, \quad (10)$$

$$\frac{1}{\alpha\beta^2\gamma} \frac{d\alpha}{dr} \frac{d\gamma}{dr} = 8\pi G \left[ +\frac{1}{\beta^2} \left( \frac{d\varphi}{dr} \right)^2 + \frac{\omega^2}{\alpha^2} \varphi^2 - m_B^2 \varphi^2 - \frac{\lambda}{2} \varphi^4 + P \right] + C, \quad (11)$$

$$\frac{1}{\alpha\beta} \frac{d}{dr} \left( \frac{1}{\beta} \frac{d\alpha}{dr} \right) = 8\pi G \left[ -\frac{1}{\beta^2} \left( \frac{d\varphi}{dr} \right)^2 + \frac{\omega^2}{\alpha^2} \varphi^2 - m_B^2 \varphi^2 - \frac{\lambda}{2} \varphi^4 + P \right] + C. \quad (12)$$

The equation of motion for the scalar field (1) becomes

$$\frac{\omega^2}{\alpha^2} \varphi + \frac{1}{\alpha\beta\gamma} \frac{d}{dr} \left( \frac{\alpha\gamma}{\beta} \frac{d\varphi}{dr} \right) - m_B^2 \varphi - \lambda \varphi^3 = 0. \quad (13)$$

The equilibrium condition is reduced to

$$\frac{dP}{dr} = -\frac{1}{\alpha} \frac{d\alpha}{dr} (P + \rho). \quad (14)$$

To simplify the equations, we make use of rescaled variables as

$$\begin{aligned}\tilde{r} &= m_B r, \quad \tilde{\varphi} = \sqrt{8\pi G} \varphi, \quad \Lambda = \frac{\lambda}{8\pi G m_B^2}, \quad \tilde{\omega} = \omega/m_B, \\ \tilde{C} &= C/m_B^2, \quad \tilde{\rho} = (8\pi G/m_B^2)\rho, \quad \text{and} \quad \tilde{P} = (8\pi G/m_B^2)P.\end{aligned}\tag{15}$$

We also rescale  $\mu$  as

$$\tilde{\mu} = \mu/m_F.\tag{16}$$

In order to study the limit of large self-interaction, we rescale the variables once more.

New variables are:

$$\begin{aligned}r_* &= \tilde{r}/\sqrt{\Lambda}, \quad \varphi_* = \sqrt{\Lambda}\tilde{\varphi}, \quad C_* = \Lambda\tilde{C}, \\ \rho_* &= \Lambda\tilde{\rho}, \quad \text{and} \quad P_* = \Lambda\tilde{P}.\end{aligned}\tag{17}$$

In the limit of large self-coupling,  $\Lambda \rightarrow \infty$ , The equations (10-14) will be reduced to

$$\frac{1}{\beta\gamma} \left( \frac{\gamma'}{\beta} \right)' = -\frac{\tilde{\omega}^2}{\alpha^2} \varphi_*^2 - \varphi_*^2 - \frac{1}{2} \varphi_*^4 - \rho_* + C_*,\tag{18}$$

$$\frac{1}{\beta^2} \frac{\alpha'}{\alpha} \frac{\gamma'}{\gamma} = +\frac{\tilde{\omega}^2}{\alpha^2} \varphi_*^2 - \varphi_*^2 - \frac{1}{2} \varphi_*^4 + P_* + C_*,\tag{19}$$

$$\frac{1}{\alpha\beta} \left( \frac{\alpha'}{\beta} \right)' = +\frac{\tilde{\omega}^2}{\alpha^2} \varphi_*^2 - \varphi_*^2 - \frac{1}{2} \varphi_*^4 + P_* + C_*,\tag{20}$$

$$\varphi_* \times \left( \frac{\tilde{\omega}^2}{\alpha^2} - 1 - \varphi_*^2 \right) = 0,\tag{21}$$

$$(\tilde{\mu}^2 - 1) \times \left( \tilde{\mu}' + \frac{\alpha'}{\alpha} \tilde{\mu} \right) = 0,\tag{22}$$

where  $'$  denotes the derivative with respect to  $r_*$ . Here we have discarded the terms of order  $1/\Lambda$  and higher. Then the terms including  $\varphi_*'$  disappear in the equations. This scheme to study the large self-coupling case was originally adopted by Colpi, Shapiro and Wasserman [4].

### III. EXACT SOLUTIONS

Now, let us solve the set of an algebraic equation and differential equations (18-22).

Eq. (21) can be solved easily. One solution for  $\varphi_*$  is

$$\varphi_*^2 = \frac{\tilde{\omega}^2}{\alpha^2} - 1. \quad (23)$$

We assume that this solution describes the configuration of the boson field for  $r_* < r_{*B}$ . Another solution for  $\varphi_*$  is

$$\varphi_*^2 = 0. \quad (24)$$

This solution is taken as the configuration of the boson field for  $r_* > r_{*B}$ . Note that  $T_{\nu}^{B\mu} = 0$  for  $r_* > r_{*B}$ . At the boundary  $r_* = r_{*B}$ ,  $\alpha$  takes the value  $\alpha = \tilde{\omega}$ .

Eq. (22) can be solved easily, too. One solution for  $\tilde{\mu}^2$  is

$$\tilde{\mu}^2 = \frac{\alpha_F^2}{\alpha^2}, \quad (25)$$

where  $\alpha_F$  is a constant. We assume that this solution describes the configuration of the fermion field for  $r_* < r_{*F}$ . Another solution for  $\tilde{\mu}^2$  is

$$\tilde{\mu}^2 = 1. \quad (26)$$

This solution is taken as the configuration of the fermion field for  $r_* > r_{*F}$ . Note that  $T_{\nu}^{F\mu} = 0$ , or  $\rho_* = P_* = 0$  for  $r_* > r_{*F}$ . At the boundary  $r_* = r_{*F}$ ,  $\alpha$  takes the value  $\alpha = \alpha_F$ .

From Eqs. (19) and (20), we find

$$\left( \frac{\alpha'}{\beta\gamma} \right)' = 0. \quad (27)$$

To solve this, we use the residual gauge choice on  $\beta$ . For *simplicity*, we take

$$\alpha\beta\gamma = Kr_*, \quad (28)$$

where  $K$  is a constant. Actually, in order to obtain the solution, one can take an arbitrary choice. This is because, as we will see later, we will solve the equations by eliminating the radial coordinate variable. Then Eq. (27) is reduced to

$$\frac{d\alpha^2}{dr_*^2} = A, \quad (29)$$

where  $A$  is a constant.

From Eqs. (19) and (28), we find<sup>1</sup>

$$\frac{1}{K^2} A \frac{\gamma \gamma'}{r_*} = \frac{1}{2} \varphi_*^4 + P_* + C_*. \quad (30)$$

Here using Eq. (29), we obtain

$$\frac{A^2}{K^2} \frac{d\gamma^2}{d\alpha^2} = \frac{1}{2} \varphi_*^4 + P_* + C_*. \quad (31)$$

Before exploring the explicit solution for the metric, we rewrite the radial line element as

$$\beta dr = \sqrt{\frac{\Lambda}{m^2}} \beta dr_* = \sqrt{\frac{\Lambda}{m^2}} \frac{K r_*}{\alpha \gamma} \frac{dr_*}{d\gamma} d\gamma = \sqrt{\frac{\Lambda}{m^2}} \frac{A}{K \alpha} \frac{1}{\frac{1}{2} \varphi_*^4 + P_* + C_*} d\gamma. \quad (32)$$

Consequently, we get

$$\beta^2 dr^2 = \frac{\Lambda}{m^2} \frac{A^2}{K^2 \alpha^2} \frac{1}{\left(\frac{1}{2} \varphi_*^4 + P_* + C_*\right)^2} d\gamma^2. \quad (33)$$

First, we solve the exterior solution for  $r_* > r_{*o}$ , where  $r_{*o} = \text{Max}[r_{*B}, r_{*F}]$ . Since the total energy momentum tensor vanishes for  $r_* > r_{*o}$ , the solution of Eq. (31) is found to be

$$\alpha^2 = \frac{A^2}{K^2 C_*^2} \left( C_* \gamma^2 - 8GM_o \right), \quad (34)$$

where  $M_o$  is an integration constant.

The full line element for the exterior solution is obtained, by substituting Eqs. (33) and (34) into (8), as

$$ds^2 = -\frac{A^2}{K^2 C_*^2} \left( CR^2 - 8GM_o \right) dt^2 + \frac{1}{CR^2 - 8GM_o} dR^2 + \frac{m^2}{\Lambda \gamma_0^2} R^2 d\theta^2, \quad (35)$$

where a new radial coordinate  $R$  is defined as

$$R = \sqrt{\frac{\Lambda}{m^2}} \gamma. \quad (36)$$

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<sup>1</sup>This equation (30) is consistent with the other equations.

We should remember here that  $C_* = C\Lambda/m^2$ . We take  $\gamma_0 = \sqrt{\frac{m^2}{\Lambda}}$  here to avoid an appearance of a conical singularity [3]. This choice makes  $R$  a usual radial coordinate when  $\theta$  varies in the range  $0 \leq \theta < 2\pi$ .

After rescaling  $\frac{A}{KC_*}dt \rightarrow dt$ , we find the metric is precisely the same as the well-known BTZ vacuum solution [7]. Thus we identify the constant  $M_o$  with the BTZ mass *of the star*.

Next we must turn to solving the interior solution of the star. We first consider the innermost region,  $r_* < r_{*i}$ , where  $r_{*i} = \text{Min}[r_{*B}, r_{*F}]$ . From Eqs. (23), (25) and (31), we obtain a differential equation:

$$\frac{A^2}{K^2} \frac{d\gamma^2}{d\alpha^2} = \frac{1}{2} \left( \frac{\tilde{\omega}^2}{\alpha^2} - 1 \right)^2 + F_* \left( \frac{\alpha_F^3}{\alpha^3} - 3 \frac{\alpha_F}{\alpha} + 2 \right) + C_*, \quad (37)$$

where  $F_* = (2G\Lambda m_F^3)/(3m_B^2)$ .

The solution of this equation is given by

$$\frac{A^2}{K^2} \left( \gamma^2 - \frac{B}{C_*} \right) = -\frac{1}{2} \frac{\tilde{\omega}^4}{\alpha^2} - \tilde{\omega}^2 \ln \frac{\alpha^2}{\tilde{\omega}^2} + \frac{1}{2} \alpha^2 + F_* \left( -2 \frac{\alpha_F^3}{\alpha} - 6\alpha_F \alpha + 2\alpha^2 + 6\alpha_F^2 \right) + C_* \alpha^2, \quad (38)$$

where  $B$  is a constant.

For the intermediate region  $r_{*i} < r < r_{*o}$ , we must consider two cases separately: for a)  $r_{*B} > r_{*F}$  and b)  $r_{*F} > r_{*B}$ .

*Case a):*  $r_{*B} > r_{*F}$

In the region  $r_{*F} < r_* < r_{*B}$ , one can solve the equation (31) with (23) and (26) and get

$$\frac{A^2}{K^2} \left( \gamma^2 - \frac{D_a}{C_*} \right) = -\frac{1}{2} \frac{\tilde{\omega}^4}{\alpha^2} - \tilde{\omega}^2 \ln \frac{\alpha^2}{\tilde{\omega}^2} + \frac{1}{2} \alpha^2 + C_* \alpha^2, \quad (39)$$

where  $D_a$  is an integration constant.

At  $r_* = r_{*F}$ , when  $\alpha = \alpha_F$ , the solutions (38) and (39) must be connected, thus we find

$$D_a = B. \quad (40)$$

The value of  $B$  is determined from the condition at the outermost boundary of the star,  $r_* = r_{*B}$ , where  $\alpha = \tilde{\omega}$ . From the condition for a smooth connection of the solutions (34) and (39), one can find



$$D_a = B = 8GM_o. \quad (41)$$

*Case b*):  $r_{*B} > r_{*F}$

In the region  $r_{*F} < r_* < r_{*B}$ , one can solve the equation (31) with (24) and (25) and get

$$\frac{A^2}{K^2} \left( \gamma^2 - \frac{D_b}{C_*} \right) = F_* \left( -2 \frac{\alpha_F^3}{\alpha} - 6\alpha_F \alpha + 2\alpha^2 + 6\alpha_F^2 \right) + C_* \alpha^2, \quad (42)$$

where  $D_b$  is an integration constant.

At  $r_* = r_{*B}$ , where  $\alpha = \tilde{\omega}$ , the solutions (38) and (42) must be connected, thus we find

$$D_b = B. \quad (43)$$

The value of  $B$  is determined from the condition at the outermost boundary of the star,  $r_* = r_{*F}$ , where  $\alpha = \alpha_F$ . From the condition for a smooth connection of the solutions (34) and (42), one can find

$$D_b = B = 8GM_o. \quad (44)$$

In both cases (*a* and *b*), we have found that the innermost solution is given by

$$\frac{A^2}{K^2} \left( \gamma^2 - \frac{8GM_o}{C_*} \right) = -\frac{1}{2} \frac{\tilde{\omega}^4}{\alpha^2} - \tilde{\omega}^2 \ln \frac{\alpha^2}{\tilde{\omega}^2} + \frac{1}{2} \alpha^2 + F_* \left( -2 \frac{\alpha_F^3}{\alpha} - 6\alpha_F \alpha + 2\alpha^2 + 6\alpha_F^2 \right) + C_* \alpha^2. \quad (45)$$

In the region  $r_{*i} < r_* < r_{*o}$ , the solutions are given by (39) and (42) with  $D_a = D_b = 8GM_o$  for *Case a*) and *b*), respectively.

In our solutions, the origin is located at  $R = 0$ , or  $\gamma = 0$ . At the center of the star, we choose the following condition:

$$\frac{K^2 \alpha_0^2}{A^2} \left( \frac{1}{2} \varphi_{*0}^4 + P_{*0} + C_* \right)^2 = 1, \quad (46)$$

where  $\alpha_0 = \alpha(\gamma = 0)$ ,  $\varphi_{*0}^2 = \varphi_*^2(\gamma = 0) = \tilde{\omega}^2/\alpha_0^2 - 1$ , and  $P_{*0} = P_*(\gamma = 0) = F_*(\alpha_F^3/\alpha_0^3 - 3\alpha_F/\alpha_0 + 2)$ .

With this choice, one can find that the limit of “no matter” fields yields the three dimensional anti-de Sitter space with no conical singularity [3].

Consequently, we obtain the relation between the BTZ mass of the boson-fermion star and  $\varphi_{*0}$  and  $\tilde{\mu}_0 = \alpha_F/\alpha_0$  from Eq. (45):

$$\frac{8GM_o}{C_*} = \frac{\frac{1}{2}\varphi_{*0}^4 + \varphi_{*0}^2 - (\varphi_{*0}^2 + 1) \ln(\varphi_{*0}^2 + 1) + 2F_*(\tilde{\mu}_0 - 1)^3 - C_*}{\left(\frac{1}{2}\varphi_{*0}^4 + F_*(\tilde{\mu}_0^3 - 3\tilde{\mu}_0 + 2) + C_*\right)^2}. \quad (47)$$

The absence of boson and fermion fields is achieved in the limit  $\varphi_{*0} = 0$  and  $\tilde{\mu}_0 = 1$ . If one define the intrinsic mass  $M_i$  as [3]

$$M_i = M_o + \frac{1}{8G}, \quad (48)$$

the intrinsic mass  $M_i$  vanishes when in the limit  $\varphi_{*0} = 0$  and  $\tilde{\mu}_0 = 1$ .

#### IV. PARTICLE NUMBERS

The particle number  $N_B$  of the boson field is given by

$$N_B = \int d^2x \sqrt{-g} \left| g^{tt} \right| i (\varphi \partial_t \varphi^* - \varphi^* \partial_t \varphi). \quad (49)$$

Therefore we can calculate the particle number of the boson by making use of the above solution. It is given by

$$\begin{aligned} N_B &= 2\pi \int \frac{2m_B \tilde{\omega}}{8\pi G \Lambda} \varphi_*^2 \frac{1}{\alpha^2} \frac{\alpha \beta \gamma}{\gamma_0} dr \\ &= \frac{m_B \tilde{\omega}}{2G \Lambda \gamma_0} \int \left( \frac{\tilde{\omega}^2}{\alpha^2} - 1 \right) \frac{1}{\alpha^2} K r_* \sqrt{\frac{\Lambda}{m_B^2}} dr_* \\ &= \frac{m_B \tilde{\omega}}{4G \Lambda \gamma_0} \sqrt{\frac{\Lambda}{m_B^2}} \int_{\alpha_0^2}^{\tilde{\omega}^2} \left( \frac{\tilde{\omega}^2}{\alpha^2} - 1 \right) \frac{1}{\alpha^2} \frac{K}{A} d\alpha^2 \\ &= \frac{m_B}{4G \Lambda \gamma_0} \sqrt{\frac{\Lambda}{m_B^2}} \frac{K \alpha_0}{A} \frac{\tilde{\omega}}{\alpha_0} \left( \frac{\tilde{\omega}^2}{\alpha_0^2} - 1 - \ln \frac{\tilde{\omega}^2}{\alpha_0^2} \right) \\ &= \frac{1}{4G m_B} \frac{\sqrt{\varphi_{*0}^2 + 1} [\varphi_{*0}^2 - \ln(\varphi_{*0}^2 + 1)]}{\frac{1}{2}\varphi_{*0}^4 + P_{*0} + C_*}. \end{aligned} \quad (50)$$

The particle number of fermions is given by

$$\begin{aligned} N_F &= \int d^2x \sqrt{-g} \frac{n_F}{\sqrt{-g_{tt}}} \\ &= 2\pi \int \frac{m_F^2}{4\pi} \left( \frac{\alpha_F^2}{\alpha^2} - 1 \right) \frac{\beta \gamma}{\gamma_0} dr \end{aligned}$$

$$\begin{aligned}
&= \frac{m_F^2}{2\gamma_0} \int \left( \frac{\alpha_F^2}{\alpha^2} - 1 \right) \frac{1}{\alpha} K r_* \sqrt{\frac{\Lambda}{m_B^2}} dr_* \\
&= \frac{m_F^2}{2\gamma_0} \sqrt{\frac{\Lambda}{m_B^2}} \int_{\alpha_0}^{\alpha_F} \left( \frac{\alpha_F^2}{\alpha^2} - 1 \right) \frac{K}{A} d\alpha \\
&= \frac{m_F^2}{2\gamma_0} \sqrt{\frac{\Lambda}{m_B^2}} \frac{K\alpha_0}{A} \left( \frac{\alpha_F}{\alpha_0} - 1 \right)^2 \\
&= \frac{m_F^2 \Lambda}{2m_B^2} \frac{(\tilde{\mu}_0 - 1)^2}{\frac{1}{2}\varphi_{*0}^4 + P_{*0} + C_*} \\
&= \frac{1}{4Gm_F} \frac{3F_*(\tilde{\mu}_0 - 1)^2}{\frac{1}{2}\varphi_{*0}^4 + P_{*0} + C_*}.
\end{aligned} \tag{51}$$

One can find that for small  $\varphi_{*0}^2$  and  $\tilde{\mu}_0 - 1$ , the mass and the particle numbers are given approximately by

$$M_i \approx \frac{\varphi_{*0}^4}{8GC_*} + \frac{3F_*}{4GC_*} (\tilde{\mu}_0 - 1)^2 \approx m_B N_B + m_F N_F. \tag{52}$$

## V. FERMION STARS

For a while, we will consider the fermion star, which can be obtained by setting  $\varphi_{*0} = 0$ . In this case, both maxima of the mass and fermion number as functions of  $\varphi_{*0}$  are located at  $\tilde{\mu}_0 = \tilde{\mu}_{0m}$ .  $\tilde{\mu}_{0m}$  is given as

$$\tilde{\mu}_{0m} - 1 = \left( \frac{2C_*}{F_*} \right)^{1/3} = \left( \frac{3C}{Gm_F^3} \right)^{1/3}. \tag{53}$$

The maximum value of  $M_i$ , which we will denote as  $M_{im}$  is found to be

$$M_{im} = \frac{1}{24G} \frac{1}{\left[ 1 + 2 \left( \frac{Gm_F^3}{3C} \right)^{1/3} \right]^2} + \frac{1}{8G}. \tag{54}$$

The maximum value of  $N_F$ , which we will denote as  $N_{Fm}$  is found to be

$$N_{Fm} = \frac{1}{2Gm_F} \frac{\left( \frac{Gm_F^3}{3C} \right)^{1/3}}{1 + 2 \left( \frac{Gm_F^3}{3C} \right)^{1/3}}. \tag{55}$$

The binding energy  $E_{bm} = M_{im} - m_F N_{Fm}$  of the fermion star with maximum fermion number is therefore given by

$$E_{bm} = -\frac{1}{6G} \frac{3 \left( \frac{Gm_F^3}{3C} \right)^{2/3} - 1}{\left[ 1 + 2 \left( \frac{Gm_F^3}{3C} \right)^{1/3} \right]^2}. \quad (56)$$

If the value of the cosmological constant is larger than  $C_{crit}$ , this state is energetically unfavorable because  $E_{bm}$  is positive. The critical value is

$$C_{crit} = \sqrt{3} G m_F^3. \quad (57)$$

It is worth noting that the value of maximum BTZ mass ( $M_{om} = M_{im} - \frac{1}{8G}$ ) is positive for all finite values of  $C$ . In particular, in the limit  $C = 0$ , the value of  $M_{om}$  vanishes. The behavior of  $M_{om}$  for small  $C$  is given by  $M_{om} \propto C^{2/3}$ . The value of  $N_{Fm}$  in the limit  $C = 0$  is  $1/(4Gm_F)$ .

If the minimum binding energy is positive, the fermion star is no longer energetically favorable. Thus when  $C > \sqrt{3} G m_F^3$ , there is no static configuration of the fermion star.

## VI. STABILITY OF BOSON-FERMION STARS

The stability of four-dimensional boson-fermion stars [5] has been studied by several authors. The perturbative stability can be judged by the line of zero eigenvalue about particle numbers in the  $(x, y)$  plane, where we denote  $\varphi_{*0}$  and  $\tilde{\mu}_0 - 1$  as  $x$  and  $y$ , respectively. The condition of zero eigenvalue is

$$\begin{vmatrix} \frac{\partial N_B}{\partial x} & \frac{\partial N_B}{\partial y} \\ \frac{\partial N_F}{\partial x} & \frac{\partial N_F}{\partial y} \end{vmatrix} = 0. \quad (58)$$

In our solutions, the condition can be reduced to

$$x^2 \left( -6C_* + 4x^2 + x^4 + 3F_* y^3 \right) + \left( 2C_* - 4x^2 - 3x^4 - F_* y^3 \right) \ln \left( 1 + x^2 \right) = 0, \quad (59)$$

where  $x \equiv \varphi_{*0}$  and  $y \equiv \tilde{\mu}_0 - 1$ .

In FIG. 1 we show the line of zero eigenvalue and contour lines of constant  $N_B$  and  $N_F$  in the  $(x, y)$  plane. The perturbatively stable solutions for boson-fermion stars lie on the

left of the line of zero eigenvalue. The solutions on the line is absolutely stable, in the sense that the stars does not exhibit radial oscillations.

In our three-dimensional model, the binding energy of the star can take positive value even in the left region of the line of zero eigenvalue. Thus the minimum of the binding energy is not always a global minimum.

The contour lines of the binding energy  $E_b = M_i - m_B N_B - m_F N_F$  is also plotted in FIG. 1.

The minimum binding energy point is given by the crossing point of zero-eigenvalue line (59) and the following line:

$$C_* y \left[ -2C_* + 2x^2 + x^4 + F_* y (6 + 3y + y^2) \right] \\ = \left[ \frac{1}{2}x^4 + F_* y^2 (3 + y) + C_* \right] \left[ 4x^2 + 3x^4 + F_* y^3 - 2(y + 2)x^2 \sqrt{1 + x^2} \right], \quad (60)$$

where  $x \equiv \varphi_{*0}$  and  $y \equiv \tilde{\mu}_0 - 1$ .

Unfortunately, the values of  $x = \varphi_{*0}$  and  $y = \tilde{\mu}_0 - 1$  which yield the configuration of the minimum binding energy cannot be obtained analytically.

The value of minimum binding energy  $E_{bm}$  is plotted as a function of  $F_*$  and  $C_*$  in FIG. 2. One can see that the minimum binding energy becomes positive for sufficiently large  $C_*$  and small  $F_*$ .

The critical line, i.e., the line of zero minimum binding energy is shown in FIG. 3. In the limit of  $F_* = 0$ , the line ends with  $C_* = C_{* \text{ crit}}$ , where  $C_{* \text{ crit}} \approx 2.5268$  is the critical value obtained in [3] for boson stars. This fact is trivial because the limit  $F_* = 0$  means no contribution of fermions.

The broken line in FIG. 3 shows the line of

$$C_* = \frac{3\sqrt{3}}{2} F_*. \quad (61)$$

It is obvious that the critical value (57) can be found for fermion stars, when the contribution of fermions is dominant.

If the minimum binding energy is positive, a boson-fermion star with any particle numbers has positive binding energy. Therefore, for arbitrary values of  $F_*$ , no stable boson-fermion star exists for sufficiently large value of  $C_*$ . The critical value is given approximately by

$$C_{* \text{ crit}}(F_*) \approx 2.5268 + \frac{3\sqrt{3}}{2}F_*. \quad (62)$$

Finally, we give the mass and particle numbers in a special limit,  $C_* \ll 1$  and  $F_* \ll 1$ . In this case, the values of  $x = \varphi_{*0}$  and  $y = \tilde{\mu}_0 - 1$  which yield the minimum binding energy can be approximately solved. These values, we denote them as  $x_m = \varphi_{*0m}$  and  $y_m = \tilde{\mu}_{0m} - 1$ , is given by

$$x_m = \varphi_{*0m} \approx \left(24C_* \left(1 - \frac{3}{2}F_*\right)\right)^{1/6}, \quad y_m = \tilde{\mu}_{0m} - 1 \approx (3C_*)^{1/3}. \quad (63)$$

These values lead to the mass  $M_{im}$ , particle number of bosons  $N_{Bm}$ , and particle number of fermions  $N_{Fm}$  of the boson-fermion star of the minimum binding energy. The approximate values of them are

$$\begin{aligned} GM_{im} &\approx \frac{1}{8} + \frac{C_*^{2/3}}{32\sqrt[3]{3} \left(\frac{3}{2}F_* + \left(1 - \frac{3}{2}F_*\right)^{2/3}\right)^2} \\ Gm_B N_{Bm} &\approx \frac{\left(1 - \frac{3}{2}F_*\right)^{2/3}}{4 \left(\frac{3}{2}F_* + \left(1 - \frac{3}{2}F_*\right)^{2/3}\right)} \\ Gm_F N_{Fm} &\approx \frac{\frac{3}{2}F_*}{4 \left(\frac{3}{2}F_* + \left(1 - \frac{3}{2}F_*\right)^{2/3}\right)}. \end{aligned} \quad (64)$$

One can see that  $Gm_B N_{Bm} + Gm_F N_{Fm} = 1/4$  in this case. It is worth noting that the BTZ mass of the boson-fermion star approaches zero in the limit of  $C_* = 0$ .

## VII. DISCUSSION

We have obtained exact solutions describing boson-fermion stars with very large self-coupling constant in  $(2+1)$  dimensions. There is a critical value for  $C_* = (\lambda/(8\pi Gm^4))C$ ,

where  $C$  is the absolute value of the (negative) cosmological constant. For  $C_* > C_{* \text{ crit}}(F_*)$  the binding energy of any solution is positive, the boson-fermion star configuration is not energetically favorable.

To study the  $(2 + 1)$  dimensional boson star, with a finite value of the self-coupling or direct couplings between bosons and fermions, we must perform numerical analyses. The present study of the exact solution is the ground for studying the properties of the boson-fermion stars in general cases.

For a finite self-coupling case, the external metric cannot be described by the exact BTZ metric. We expect that the external metric will approach the black hole metric in the large distance asymptotically if the global attractive force due to the cosmological constant is sufficiently effective. On the other hand, another types of the external metric will also be expected, which exhibit different behaviors from the BTZ metric. The possibility is worth studying.

In the present paper, we have discussed only static configurations. We are interested also in the process of formation of the boson-fermion stars. The formation of stars or black holes (or possible naked singularities) will be clarified by the investigation of time-dependent processes. The quantum mechanical processes, including even the tunneling process, may play an important role in the creation of small boson-fermion stars. This subject will be considered in a separate research in the future.

The time-dependent solution describing radial pulsations in  $(2 + 1)$  dimensions can be studied first of all. It will be discussed in elsewhere.

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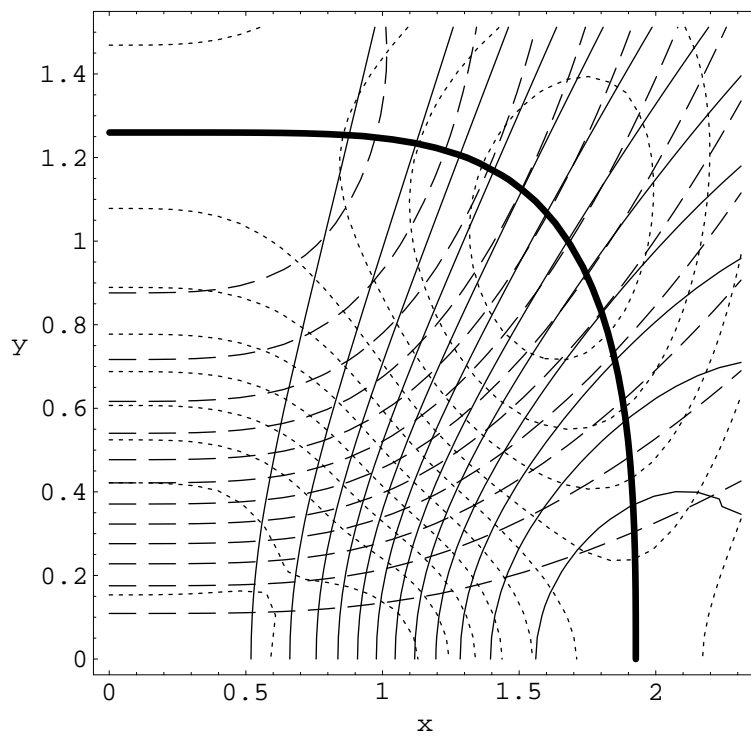
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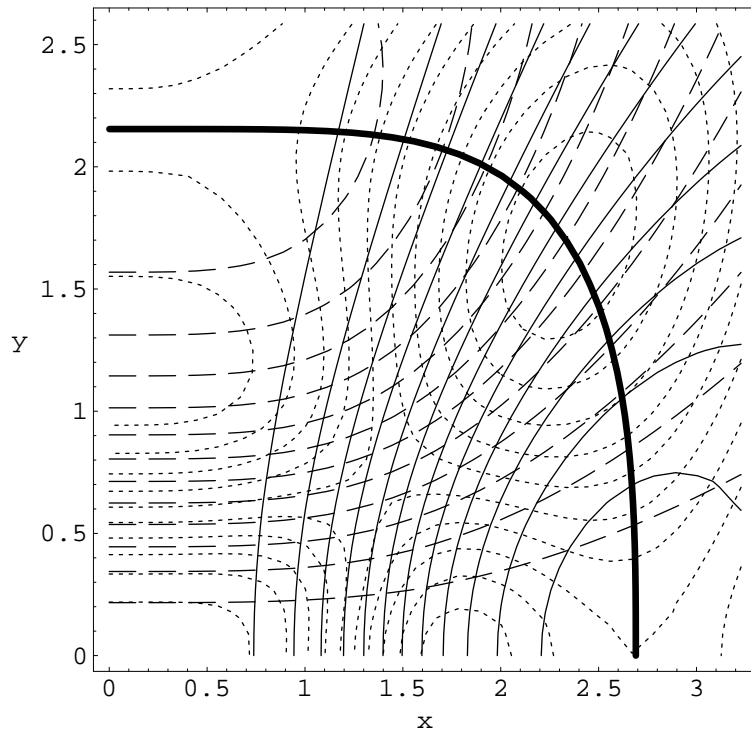


# FIGURES

-a-



-b-



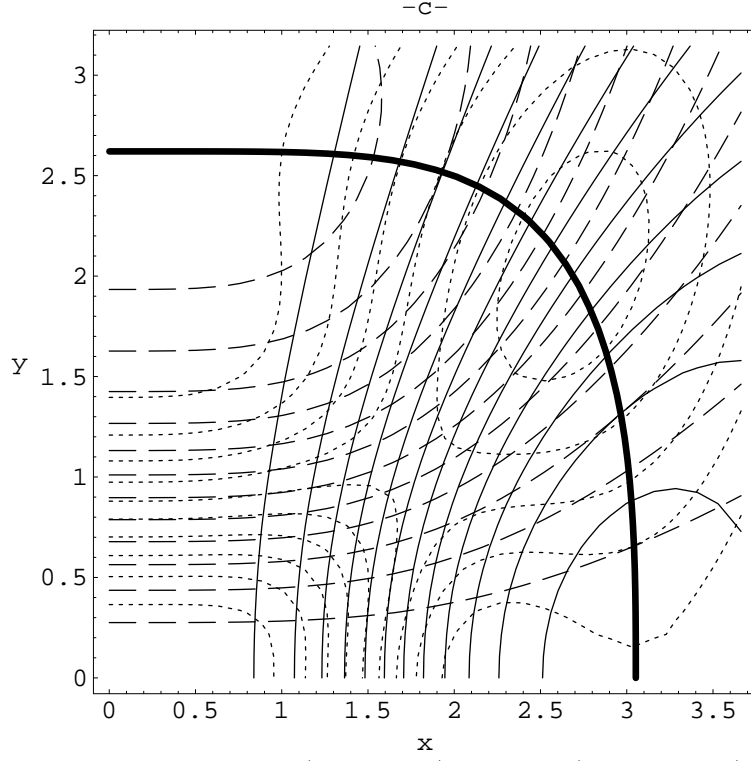


FIG. 1. Contour lines of constant  $N_B$  (solid lines) and  $N_F$  (broken lines) in the  $(x, y)$  plane, where  $x = \varphi_{*0}$  and  $y = \tilde{\mu}_0 - 1$ . The bold line shows the line of zero eigenvalue (see text). Contour lines of the binding energy is also shown by dotted lines. In these figures,  $F_*$  is set to unity, and  $C_*$  is (a) 1, (b) 5, and (c) 9.

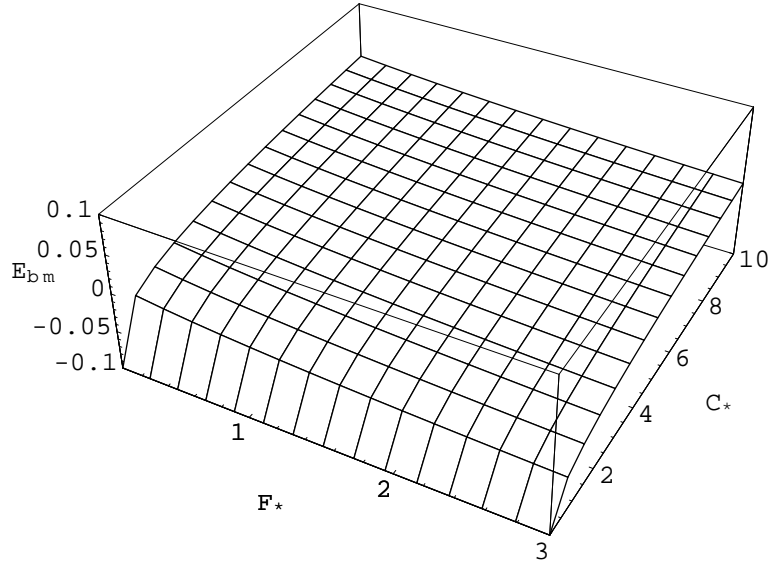


FIG. 2. The value of minimum binding energy of a boson-fermion star as a function of  $F_*$  and  $C_*$ .

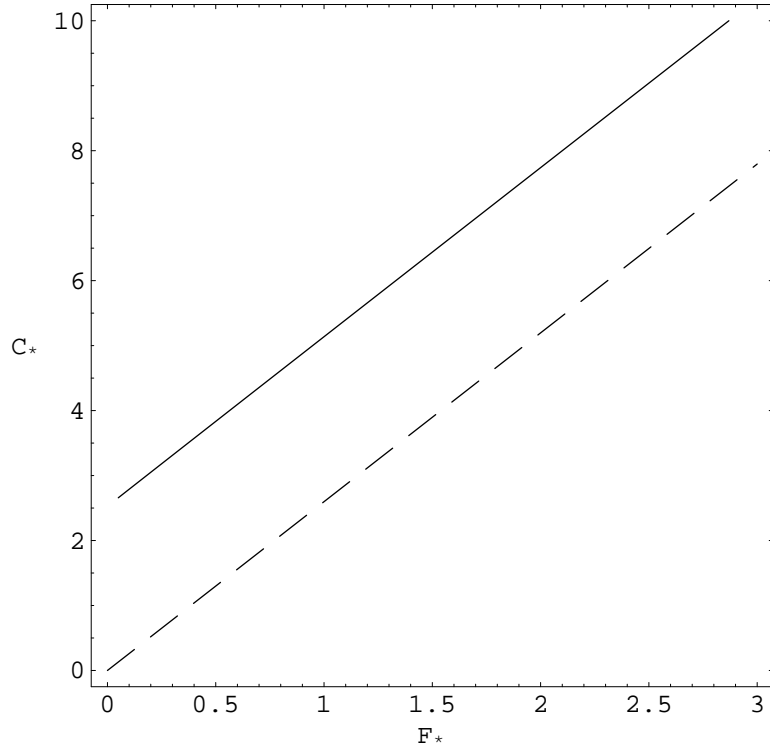


FIG. 3. The line of zero minimum binding energy in the  $(F_*, C_*)$  plane (the solid line). The broken line shows the line of  $C_* = \frac{3\sqrt{3}}{2}F_*$ .